Section de mathématiques

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Functional analysis 2
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Sheet 9

Exercises for 'Functional Analysis 2' [MATH-404]

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Ex 9.1 (The chain rule on Banach spaces)
Let X, Y, Z be Banach spaces and $U \subset X$, $V \subset Y$ be open. Assume that $F : U \to Y$ is differentiable in $x_0 \in U$ with $F(x_0) \in V$ and that $G : V \to Z$ is differentiable in $F(x_0)$. Show that $G \circ F : U \to Z$ is differentiable in x_0 with

$$(G \circ F)'(x_0) = G'(F(x_0))F'(x_0).$$

Hint: Using little-o notation simplifies the calculations.

Solution 9.1 : Using little-o notation, we have for $h \in X$ with $x_0 + h \in U$ and $y \in Y$ with $F(x_0) + y \in V$ that

$$\begin{aligned} F(x_0 + h) &= F(x_0) + F'(x_0)h + o(\|h\|_X), \\ G(F(x_0) + y) &= G(F(x_0)) + G'(F(x_0))y + o(\|y\|_Y). \end{aligned}$$

For $\|h\|_X$ small enough we can apply these equations for $y = F'(x_0)h + o(\|h\|_X)$, so that

$$\begin{aligned} G(F(x_0 + h)) &= G(F(x_0) + F'(x_0)h + o(\|h\|_X)) \\ &= G(F(x_0)) + G'(F(x_0))(F'(x_0)h + o(\|h\|_X)) + o(\|F'(x_0)h + o(\|h\|_X)\|_Y). \end{aligned}$$

Since $F'(x_0)$ and $G'(F(x_0))$ are bounded, linear operators, it holds that $G'(F(x_0))(o(\|h\|_X)) = o(\|h\|_X)$ (in Z) and $o(\|F'(x_0)h + o(\|h\|_X)\|_Y) = o(\|h\|_X)$ (also in Z). Hence

$$G(F(x_0 + h)) = G(F(x_0)) + G'(F(x_0))F'(x_0)h + o(\|h\|_X),$$

which proves the claim.

Ex 9.2 (Fréchet-differentiability)
a) Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-function. Show that

$$u \mapsto F(u) = \int_0^1 f(u(x))\, \mathrm{d}x$$

is differentiable on $C([0,1])$ equipped with the maximum norm and compute its derivative. Is the derivative continuous ?

b) Let $k \in C([0,1] \times [0,1])$, $f \in C^1([0,1] \times \mathbb{R})$ and consider the **Hammerstein operator**

$$u \mapsto F(u) = \int_0^1 k(\,\cdot\,, x) f(x, u(x))\, \mathrm{d}x.$$

Check that $F\colon C([0,1]) \to C([0,1])$ is continuously differentiable.

Solution 9.2 :

a) Fix $u \in C([0,1])$. We claim that the derivative of F at u is given by

$$F'(u)v = \int_0^1 f'(u(x))v(x)\,\mathrm{d}x.$$

First note that this functional is indeed linear in v and bounded since

$$|F'(u)v| \le \left(\int_0^1 |f'(u(x))|\,\mathrm{d}x\right) \|v\|_{C([0,1])}$$

and the integral above is finite since by continuity u is bounded and f' is bounded on compact subsets of $\mathbb{R}$. Next, we show that $F'(u)$ is indeed the derivative of F at u. Let $h \in C([0,1])$. By the mean value theorem for real-valued functions we deduce that

$$\begin{aligned}|f(u(x)+h(x)) - f(u(x)) - f'(u(x))h(x)| &\le \sup_{t\in[0,1]} |f'(u(x)+th(x)) - f'(u(x))| \cdot |h(x)| \\ &\le \sup_{t\in[0,1]} |f'(u(x)+th(x)) - f'(u(x))| \cdot \|h\|_{C([0,1])}.\end{aligned}$$

Now consider a sequence $h_n \in C([0,1])$ such that $h_n \to 0$. Then by the above estimate

$$\begin{aligned}\frac{|F(u+h_n) - F(u) - F'(u)h_n|}{\|h_n\|_{C([0,1])}} &\le \frac{1}{\|h_n\|_{C([0,1])}} \int_0^1 |f(u(x)+h_n(x)) - f(u(x)) - f'(u(x))h_n(x)|\,\mathrm{d}x \\ &\le \int_0^1 \sup_{t\in[0,1]} |f'(u(x)+th_n(x)) - f'(u(x))|\,\mathrm{d}x.\end{aligned}$$

Since $\delta_n := \max_{x\in[0,1]} |h_n(x)|$ converges to 0 as $n \to +\infty$ and $u_0 := \max_{x\in[0,1]} |u(x)| < +\infty$, it follows that

$$\int_0^1 \sup_{t\in[0,1]} |f'(u(x)+th_n(x)) - f'(u(x))|\,\mathrm{d}x \le \sup_{u\in[-u_0,u_0]} \sup_{0\le\delta<\delta_n} |f'(u+\delta) - f'(u)|.$$

Note that f' is uniformly continuous on every compact interval of $\mathbb{R}$. Hence the last term tends to 0 as $n \to +\infty$. This shows the differentiability of F.

Yes, the derivative $F'(u)$ depends continuously on u; let us show it. Let $u_n \in C([0,1])$ such that $u_n \to u$ uniformly on $[0,1]$. Then for every $v \in C([0,1])$ with $\|v\|_{C([0,1])} = 1$ we have that

$$|F'(u_n)v - F'(u)v| \le \int_0^1 |f'(u_n(x)) - f'(u(x))||v(x)|\,\mathrm{d}x \le \int_0^1 |f'(u_n(x)) - f'(u(x))|\,\mathrm{d}x$$

The last term is independent of v, so that taking the supremum over such functions v we deduce that

$$\|F'(u_n) - F'(u)\|_{C([0,1])'} \le \int_0^1 |f'(u_n(x)) - f'(u(x))|\,\mathrm{d}x.$$

The last term tends to zero by the dominated convergence theorem since the $(u_n)_n$ are uniformly bounded, and f' is bounded on compact sets. Hence $F \in C^1(C([0,1]),\mathbb{R})$.

b) Fix $u \in C([0,1])$ and a direction $v \in C([0,1])$. Since for every $y \in [0,1]$ we have

$$\frac{1}{\varepsilon}\big[F(u+\varepsilon v)(y) - F(u)(y)\big] = \int_0^1 k(y,x)\frac{f\big(x,u(x)+\varepsilon v(x)\big) - f(x,u(x))}{\varepsilon}\,\mathrm{d}x$$

and the function under the integral converges (pointwise) to $\partial_y f(x, u(x))\, v(x)$ for $\varepsilon \to 0$ (where $\partial_y f$ denotes the partial derivative of $f(x, y)$ w.r.t. y), we expect the derivative of F to be given by

$$F'(u)v = \int_0^1 k(\,\cdot\,, x)\, \partial_y f(x, u(x))\, v(x)\, \mathrm{d}x.$$

Both functions k and $\partial_y f$ are continuous, thus

$$\begin{aligned} \|F'(u)v\|_{C([0,1])} &\le \int_0^1 \|k(\,\cdot\,, x)\|_{C([0,1])}\, |\partial_y f(x, u(x))\, v(x)|\, \mathrm{d}x \\ &\le \Big(\int_0^1 \|k(\,\cdot\,, x)\|_{C([0,1])}\, |\partial_y f(x, u(x))|\, \mathrm{d}x \Big) \|v\|_{C([0,1])} \end{aligned}$$

so the operator $F'(u)$ is bounded, in addition to being obviously linear.

To prove that $F'(u)$ is indeed the Fréchet derivative of F at u, note that for all $v \in C([0,1])$

$$\begin{aligned} &\|F(u+v) - F(u) - F'(u)v\|_{C([0,1])} \\ &\qquad \le \int_0^1 \|k(\,\cdot\,, x)\|_{C([0,1])}\, |f(x, u(x) + v(x)) - f(x, u(x)) - \partial_y f(x, u(x))v(x)|\, \mathrm{d}x \end{aligned}$$

and by using the mean value theorem for the function $y \mapsto f(x, y)$

$$\begin{aligned} |f(x, u(x) + v(x)) - &f(x, u(x)) - \partial_y f(x, u(x))v(x)| \\ &\le \sup_{0 \le t \le 1} |\partial_y f(x, u(x) + tv(x)) - \partial_y f(x, u(x))|\, |v(x)| \\ &\le \sup_{0 \le t \le 1} |\partial_y f(x, u(x) + tv(x)) - \partial_y f(x, u(x))|\, \|v\|_{C([0,1])}. \end{aligned}$$

Since $\partial_y f$ is uniformly continuous on $[0,1] \times [-\|u\|_{C([0,1])} - 1, \|u\|_{C([0,1])} + 1]$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|z| < \delta$ implies that

$$|\partial_y f(x, y + z) - \partial_y f(x, y)| < \varepsilon$$

whenever $x \in [0,1]$, $|y| \le \|u\|_{C([0,1])}$.
Thus $\|v\|_{C([0,1])} < \delta$ implies

$$\sup_{0 \le t \le 1} |\partial_y f(x, u(x) + tv(x)) - \partial_y f(x, u(x))| < \varepsilon$$

and so

$$\frac{\|F(u+v) - F(u) - F'(u)v\|_{C([0,1])}}{\|v\|_{C([0,1])}} \le \Big(\int_0^1 \|k(\,\cdot\,, x)\|_{C([0,1])}\, \mathrm{d}x \Big) \varepsilon$$

for $\|v\|_{C([0,1])} \le \varepsilon$. Since ε was arbitrary, this completes the proof.

Finally, to see that $u \mapsto F'(u)$ is continuous, note that for any $u_1, u_2 \in C([0,1])$

$$\begin{aligned} \|F'(u_1) - F'(u_2)\|_{\mathcal{L}(C([0,1]))} &= \sup_{v : \|v\|_{C([0,1])} \le 1} \|F'(u_1)v - F'(u_2)v\|_{C([0,1])} \\ &\le \int_0^1 \|k(\,\cdot\,, x)\|_{C([0,1])} |f(x, u_1(x)) - f(x, u_2(x))|\, \mathrm{d}x. \end{aligned}$$

Therefore, the mean value theorem and the same reasoning used to show differentiability can be employed to demonstrate the continuity.

Ex 9.3 (Gâteaux=Fréchet for Lipschitz functions on finite-dimensional spaces*)
Suppose $F\colon X \to Y$ is a Lipschitz function from a finite-dimensional Banach space X to

a (possibly infinite-dimensional) Banach space Y. Prove that if F is Gâteaux-differentiable at some point x, then it is also Fréchet-differentiable at that point.

Ex 9.4 (Gâteaux-differentiability)

a) We saw in the lecture that the function $F : L^2([0,1]) \to L^2([0,1])$ defined by $F(u)(x) = \cos(u(x))$ is not differentiable in 0 (actually it is nowhere differentiable). Prove that it is Gâteaux-differentiable on $L^2([0,1])$ with $\delta F(u)v = -(\sin \circ u) \cdot v$.

b) Show that if the function $u_0 \in C([0,1])$ is such that $|u_0|$ attains its maximum on $[0,1]$ at a single point t_0, the norm $u \mapsto \|u\| = \sup_{x\in[0,1]} |u(x)|$ on the Banach space $C([0,1])$ is Gâteaux differentiable at u_0 and for any direction $v \in C([0,1])$

$$\delta\|u_0\|v = v(t_0) \cdot \operatorname{sign} u_0(t_0).$$

c)* Show that the norm is not Gâteaux differentiable at u_0 if u_0 does not satisfy the above assumption.

Hint: Show that for any maximizer t_0 of $|u(t)|$ the map found in b) has to coincide with the Gâteaux-derivative.

Solution 9.4 :

a) Fix $u \in L^2([0,1])$ and $v \in L^2([0,1])$ such that $\|v\|_{L^2([0,1])} = 1$. Then for any $\varepsilon > 0$, the mean value theorem for real valued functions yields that

$$|\cos(u(x) + \varepsilon v(x)) - \cos(u(x)) + \sin(u(x))\varepsilon v(x)| \leq \varepsilon |v(x)| \sup_{|\xi|<\varepsilon|v(x)|} |\sin(u(x)) - \sin(u(x) + \xi)|$$

Integrating the square of this inequality over $[0,1]$, we find that

$$\left\|\frac{F(u+\varepsilon v) - F(u)}{\varepsilon} + (\sin \circ u)v\right\|^2_{L^2([0,1])} \leq \int_0^1 |v(x)|^2 \sup_{|\xi|<\varepsilon|v(x)|} |\sin(u(x)) - \sin(u(x)+\xi)|^2 \, dx$$

We argue that the last term vanishes when $\varepsilon \to 0$. Since $v(x)$ is finite almost everywhere, the integrand converges to 0 as $\varepsilon \to 0$ for almost every $x \in [0,1]$. Moreover, as $|\sin(x)| \leq 1$ on the real line, we can bound the integrand by $4|v(x)|^2$, which is integrable and independent of ε. Thus we can apply the dominated convergence theorem and conclude the proof.

b) Let $v \in C([0,1])$ be a fixed direction. We need to study the limit

$$\lim_{\varepsilon\to 0} \frac{\|u_0 + \varepsilon v\| - \|u_0\|}{\varepsilon}.$$

We have $\|u_0\| = |u_0(t_0)|$ and denoting $t_\varepsilon \in [0,1]$ a point where $u_0 + \varepsilon v$ attains its maximum we also have

$$\|u_0 + \varepsilon v\| = |u_0(t_\varepsilon) + \varepsilon v(t_\varepsilon)| \geq |u_0(t_0) + \varepsilon v(t_0)|.$$

As a consequence,

$$\begin{aligned} 0 < |u_0(t_0)| - |u_0(t_\varepsilon)| &\leq |u_0(t_0) + \varepsilon v(t_0)| + |\varepsilon v(t_0)| - |u_0(t_\varepsilon)| \\ &\leq |u_0(t_\varepsilon) + \varepsilon v(t_\varepsilon)| + |\varepsilon v(t_0)| - |u_0(t_\varepsilon)| \leq |\varepsilon|\big(|v(t_\varepsilon)| + |v(t_0)|\big), \end{aligned}$$

thus we get $\lim_{\varepsilon\to 0} |u_0(t_\varepsilon)| = |u_0(t_0)|$. Since $[0,1]$ is compact, we can assume that t_ε converges to some $t^* \in [0,1]$. Because $|u_0(t_0)| > |u_0(t)|$ for any $t \neq t_0$, it follows that $t^* = t_0$, so we have

$$\lim_{\varepsilon\to 0} t_\varepsilon = t_0.$$

Assume that $u_0(t_0) > 0$. Then, for all ε small enough, we have

$$|u_0(t_\varepsilon) + \varepsilon v(t_\varepsilon)| - |u_0(t_0)| = u_0(t_\varepsilon) + \varepsilon v(t_\varepsilon) - u_0(t_0) \leq \varepsilon v(t_\varepsilon)$$

and

$$|u_0(t_0) + \varepsilon v(t_0)| - |u_0(t_0)| = u_0(t_0) + \varepsilon v(t_0) - u_0(t_0) = \varepsilon v(t_0)$$

therefore

$$\varepsilon v(t_0) \leq |u_0(t_\varepsilon) + \varepsilon v(t_\varepsilon)| - |u_0(t_0)| \leq \varepsilon v(t_\varepsilon)$$

which gives

$$\lim_{\varepsilon \to 0} \frac{\|u_0 + \varepsilon v\| - \|u_0\|}{\varepsilon} = v(t_0).$$

c) To show that the condition is necessary, suppose that there exist distinct $t_0, t_1 \in [0,1]$ such that

$$|u_0(t_0)| = |u_0(t_1)| \geq |u_0(t)|, \quad \text{for all } t \in [0,1].$$

Let $v \in C([0,1])$. Note that we have

$$\|u\| + \varepsilon v(t_0)\operatorname{sgn}(u(t_0)) = (u(t_0) + \varepsilon v(t_0))\operatorname{sgn}(u(t_0)) \leq \|u + \varepsilon v\|$$

If the norm were Gâteaux-differentiable in u, this would imply that

$$\delta\|u\|v \geq v(t_0)\operatorname{sgn}(u(t_0)).$$

By linearity of the derivative and arbitrariness of v, this can only hold if there is equality, so that

$$\delta\|u\|v = v(t_0)\operatorname{sgn}(u(t_0)).$$

But by the same argument we find that

$$\delta\|u\|v = v(t_1)\operatorname{sgn}(u(t_1)).$$

This cannot hold for all $v \in C([0,1])$, so that the norm is not Gâteaux-differentiable in u_0.